# 2018 AIME II Problems

## Problem 1

Points , , and lie in that order along a straight path where the distance from to is meters. Ina runs twice as fast as Eve, and Paul runs twice as fast as Ina. The three runners start running at the same time with Ina starting at and running toward , Paul starting at and running toward , and Eve starting at and running toward . When Paul meets Eve, he turns around and runs toward . Paul and Ina both arrive at at the same time. Find the number of meters from to .

Solution 1

We know that in the same amount of time given, Ina will run twice the distance of Eve, and Paul would run quadruple the distance of Eve. Let's consider the time it takes for Paul to meet Eve: Paul would've run 4 times the distance of Eve, which we can denote as . Thus, the distance between and is . In that given time, Ina would've run twice the distance , or units. Now, when Paul starts running back towards , the same amount of time would pass since he will meet Ina at his starting point. Thus, we know that he travels another units and Ina travels another units. Therefore, drawing out the diagram, we find that , and distance between and is the distance Ina traveled, or

Solution 2

Let be the distance from to . Then the distance from to is . Since Eve is the slowest, we can call her speed , so that Ina's speed is and Paul's speed is . For Paul and Eve to meet, they must cover a total distance of which takes them a time of . Paul must run the same distance back to , so his total time is . For Ina to reach , she must run a distance of at a speed of , taking her a time of . Since Paul and Ina reach at the same time, we know that (notice that cancels out on both sides). Solving the equation gives .

## Problem 2

Let , , and , and for define recursively to be the remainder when ( ) is divided by . Find • • .

Solution When given a sequence problem, one good thing to do is to check if the sequence repeats itself or if there is a pattern. After computing more values of the sequence, it can be observed that the sequence repeats itself every 10 terms starting at . , , , , , , , , , , , , , We can simplify the expression we need to solve to • • . Our answer is • • .

## Problem 3

Find the sum of all positive integers such that the base- integer is a perfect square and the base- integer is a perfect cube.

Solution

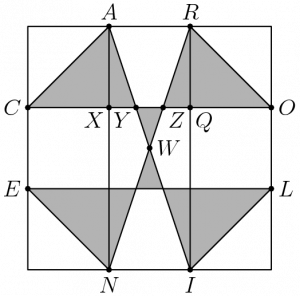
The first step is to convert and into base-10 numbers. Then, we can write and . It should also be noted that . Because there are less perfect cubes than perfect squares for the restriction we are given on , it is best to list out all the perfect cubes. Since the maximum can be is 1000 and • , we can list all the perfect cubes less than 2007. Now, must be one of . However, will always be odd, so we can eliminate the cubes of the even numbers and change our list of potential cubes to , and . Because is a perfect square and is clearly divisible by 3, it must be divisible by 9, so is divisible by 3. Thus the cube, which is , must also be divisible by 3. Therefore, the only cubes that could potentially be now are and . We need to test both of these cubes to make sure is a perfect square. If we set equal to , . If we plug this value of b into , the expression equals , which is indeed a perfect square. If we set equal to , . If we plug this value of b into , the expression equals , which is . We have proven that both and are the only solutions, so .

## Problem 4

In equiangular octagon , and . The self-intersecting octagon encloses six non-overlapping triangular regions. Let be the area enclosed by , that is, the total area of the six triangular regions. Then , where and are relatively prime positive integers. Find .

Solution

We can draw and introduce some points.



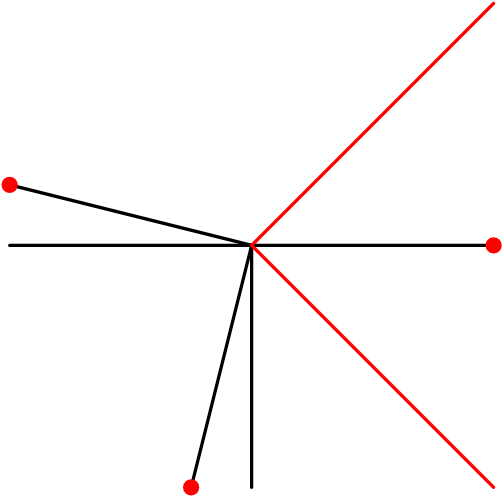
The diagram is essentially a 3x3 grid where each of the 9 squares making up the grid have a side length of 1. In order to find the area of , we need to find 4 times the area of and 2 times the area of . Using similar triangles and , . Therefore, the area of is Since and , and . Therefore, the area of is Our final answer is

## Problem 5

Suppose that , , and are complex numbers such that , , and , where . Then there are real numbers and such that . Find .

Solution 1

First we evaluate the magnitudes. , , and . Therefore, , or . Divide to find that , , and .



This allows us to see that the argument of is , and the argument of is . We need to convert the polar form to a standard form. Simple trig identities show and . More division is needed to find what is. Written by a1b2

Solution 2

Dividing the first equation by the second equation given, we find that . Substituting this into the third equation, we get . Taking the square root of this is equivalent to halving the argument and taking the square root of the magnitude. Furthermore, the second equation given tells us that the argument of is the negative of that of , and their magnitudes multiply to . Thus we have and . To find , we can use the previous substitution we made to find that Therefore,

Solution 3

We are given that . Thus . We are also given that . Thus . We are also given that = . Substitute and into = . We have . Multiplying out we get . Thus . Simplifying this fraction we get . Cross-multiplying the fractions we get or . Now we can rewrite this as . Let .Thus or . We can see that and thus or .We also can see that because there is no real term in . Thus or . Using the two equations and we solve by doing system of equations that and . And so . Because , then . Simplifying this fraction we get or . Multiplying by the conjugate of the denominator () in the numerator and the denominator and we get . Simplifying this fraction we get . Given that = we can substitute We can solve for z and get . Now we know what , , and are, so all we have to do is plug and chug. or Now or . Thus is our final answer.(David Camacho)

Solution 4

We observe that by multiplying and we get Next, we divide by to get We have We can write in the form of so we get Then, and Solving this system of equations is relatively simple. We have two cases, and Case 1: so We solve for and by plugging in to the two equations. We see and so and Solving, we end up with as our answer. Case 2: so Again, we solve for and We find so We again have Solution 5

(Based on advanced mathematical knowledge) According to the Euler's Theory, we can rewrite , and as As a result, Also, it is clear that So , or Also, we have So now we have , , , and . Solve these above, we get So we can get Use we can find that So So we have and . As a result, we finally get

## Problem 6

A real number is chosen randomly and uniformly from the interval . The probability that the roots of the polynomial are all real can be written in the form , where and are relatively prime positive integers. Find .

Solution

The polynomial we are given is rather complicated, so we could use Rational Root Theorem to turn the given polynomial into a degree-2 polynomial. With Rational Root Theorem, are all possible rational roots. Upon plugging these roots into the polynomial, and make the polynomial equal 0 and thus, they are roots that we can factor out. The polynomial becomes: Since we know and are real numbers, we only need to focus on the quadratic. We should set the discriminant of the quadratic greater than or equal to 0. . This simplifies to: or This means that the interval is the "bad" interval. The length of the interval where can be chosen from is 38 units long, while the bad interval is 2 units long. Therefore, the good interval is 36 units long.

## Problem 7

Triangle has side lengths , , and . Points are on segment with between and for , and points are on segment with between and for . Furthermore, each segment , , is parallel to . The segments cut the triangle into regions, consisting of trapezoids and triangle. Each of the regions has the same area. Find the number of segments , , that have rational length.

Solution 1

For each between and , the area of the trapezoid with as its bottom base is the difference between the areas of two triangles, both similar to . Let be the length of segment . The area of the trapezoid with bases and is times the area of . (This logic also applies to the topmost triangle if we notice that .) However, we also know that the area of each shape is times the area of . We then have . Simplifying, . However, we know that , so , and in general, and . The smallest that gives a rational is , so is rational if and only if for some integer .The largest such that is less than is , so has possible values.

Solution 2

We have that there are trapezoids and triangle of equal area, with that one triangle being . Notice, if we "stack" the trapezoids on top of the way they already are, we'd create a similar triangle, all of which are similar to , and since the trapezoids and have equal area, each of these similar triangles, have area , and so . We want the ratio of the side lengths . Since area is a 2-dimensional unit of measurement, and side lengths are 1-dimensional, the ratio is simply the square root of the areas, or , so there are solutions.

Solution 3

Let stand for , and . All triangles are similar by AAA. Let the area of be . The next trapezoid will also have an area of , as given. Therefore, has an area of . The ratio of the areas is equal to the square of the scale factor for any plane figure and its image. Therefore, , and the same if is substituted for throughout. We want the side to be rational. Setting up proportions: which shows that . In order for to be rational, must be some rational multiple of . This is achieved at . We end there as . There are 20 numbers from 1 to 20, so there are solutions.

## Problem 8

A frog is positioned at the origin of the coordinate plane. From the point , the frog can jump to any of the points , , , or . Find the number of distinct sequences of jumps in which the frog begins at and ends at .

Solution 1

We solve this problem by working backwards. Notice, the only points the frog can be on to jump to in one move are and . This applies to any other point, thus we can work our way from to , recording down the number of ways to get to each point recursively. , , , A diagram of the numbers: 5 - 20 - 71 - 207 - 3 - 10 - 32 - 84 - 207 2 - 5 - 14 - 32 - 71 1 - 2 - 5 - 10 - 20 1 - 1 - 2 - 3 - 5

Solution 2

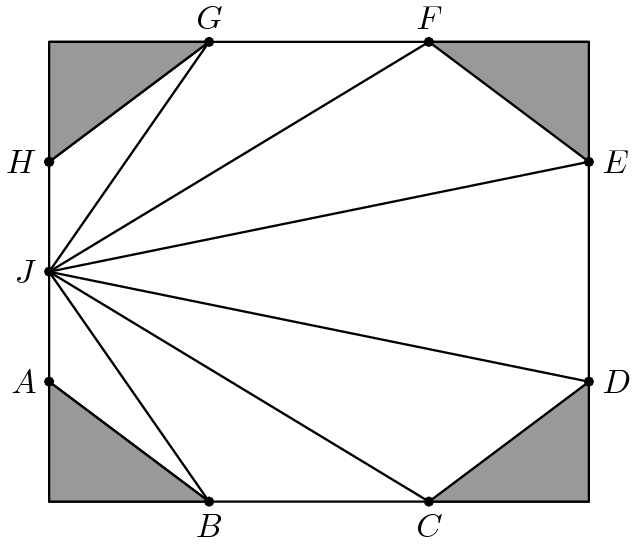
We'll refer to the moves , , , and as , , , and , respectively. Then the possible sequences of moves that will take the frog from to are all the permutations of , , , , , , , , and . We can reduce the number of cases using symmetry. Case 1: There are possibilities for this case. Case 2: or There are possibilities for this case. Case 3: There are possibilities for this case. Case 4: or There are possibilities for this case. Case 5: or There are possibilities for this case. Case 6: There are possibilities for this case. Adding up all these cases gives us ways.

Solution 3

(General Case) Mark the total number of distinct sequences of jumps for the frog to reach the point as . Consider for each point in the first quadrant, there are only possible points in the first quadrant for frog to reach point , and these points are . As a result, the way to count is Also, for special cases, Start with , use this method and draw the figure below, we can finally get (In order to make the LaTeX thing more beautiful to look at, I put to make every number a -digits number) So the total number of distinct sequences of jumps for the frog to reach is .

## Problem 9

Octagon with side lengths and is formed by removing 6-8-10 triangles from the corners of a rectangle with side on a short side of the rectangle, as shown. Let be the midpoint of , and partition the octagon into 7 triangles by drawing segments , , , , , and . Find the area of the convex polygon whose vertices are the centroids of these 7 triangles.



Solution 1

(Massive Shoelace) We represent Octagon in the coordinate plane with the upper left corner of the rectangle being the origin. Then it follows that . Recall that the centroid is way up each median in the triangle. Thus, we can find the centroids easily by finding the midpoint of the side opposite of point . Furthermore, we can take advantage of the reflective symmetry across the line parallel to going through by dealing with less coordinates and ommiting the in the shoelace formula. By doing some basic algebra, we find that the coordinates of the centroids of are and , respectively. We'll have to throw in the projection of the centroid of to the line of reflection to apply shoelace, and that point is Finally, applying Shoelace, we get:

Solution 2

(Homothety) Draw the heptagon whose vertices are the midpoints of octagon except . Note that passes through corresponding vertices of the two heptagons. Also, by centroid properties, our ratio between the sidelengths is , and their area ratio is hence . So, we have a homothety. We now compute the area of our large heptagon. We can divide this into two congruent trapezoids and a triangle. The area of each trapezoid is . The area of each triangle is . Hence, the area of the large heptagon is . Then from our homothety, the area of the required heptagon is

## Problem 10

Find the number of functions from to that satisfy for all in .

Solution 1

We do casework on the number of fixed points of , that is, the number of such that . We know that at least one such exists, namely .

There are five ways to choose the fixed point. WLOG let the fixed point be . Then at least one of must map onto , the only fixed point. Suppose exactly one of these values maps to ; there are four ways to choose such a value. WLOG let it be . Then all of must map to in order to be mapped to in the next iteration. There are solutions in this case. Suppose exactly two of these values map to ; there are ways to choose such values. WLOG let them be and . Then and must map to one of and , where there are ways to do so. Therefore there are solutions in this case. Suppose exactly three of these values map to ; there are ways to choose such values. WLOG let them be , , and . Then must map to one of , , and , where there are solutions. Therefore there are solutions in this case. Suppose exactly four of these values map to . Then everything maps to and there is solution in this case. Therefore there are solutions in Case 1.

There are ways to choose the fixed points. WLOG let them be and . Then at least one of must map onto or . Suppose exactly one of these values maps to or ; there are three ways to choose this value, and two ways to choose the value it maps to. WLOG let it be . Then both and must map to , for a total of solutions in this case. Suppose exactly two of these values map to or ; there are ways to choose these values, and ways to choose the values they map to. WLOG let them be and . Then must map to one of and , for two possible choices. Therefore there are solutions in this case. Suppose exactly three of these values map to or ; then everything maps to or and there are solutions in this case. Therefore there are solutions in Case 2.

There are ways to choose the fixed points. WLOG let them be , , and . Then at least one of and must map onto , , or . Suppose exactly one of these values map to , , or ; there are two ways to choose this value, and three ways to choose the value is maps to. WLOG let it be . Then must map to , for a total of solutions in this case. Suppose exactly two of these values map to , , or ; then everything maps to , , or , and there are solutions in this case. Therefore there are solutions in Case 3.

There are ways to choose the fixed points. WLOG let them to , , , and . Then must map to one of these values, for a total of solutions in Case 4.

Since everything is a fixed point, there is only one solution in Case 5. Therefore there are a total of functions that satisfy the problem condition.

Solution 2

We can do some caseworks about the special points of functions for . Let , and be three different elements in set . There must be elements such like in set satisfies , and we call the points such like on functions are "Good Points" (Actually its academic name is "fixed-points"). The only thing we need to consider is the "steps" to get "Good Points". Notice that the "steps" must less than because the highest iterations of function is . Now we can classify cases of “Good points” of .

One "step" to "Good Points": Assume that , then we get , and , so .

Two "steps" to "Good Points": Assume that and , then we get , and , so .

Three "steps" to "Good Points": Assume that , and , then we get , and , so . Divide set into three parts which satisfy these three cases, respectively. Let the first part has elements, the second part has elements and the third part has elements, it is easy to see that . First, there are ways to select for Case 1. Second, we have ways to select for Case 2. After that we map all elements that satisfy Case 2 to Case 1, and the total number of ways of this operation is . Finally, we map all the elements that satisfy Case 3 to Case 2, and the total number of ways of this operation is . As a result, the number of such functions can be represented in an algebraic expression contains , and : Now it's time to consider about the different values of , and and the total number of functions satisfy these values of , and : For , and , the number of is For , and , the number of is For , and , the number of is For , and , the number of is For , and , the number of is For , and , the number of is For , and , the number of is For , and , the number of is For , and , the number of is For , and , the number of is For , and , the number of is Finally, we get the total number of function , the number is

## Problem 11

Find the number of permutations of such that for each with , at least one of the first terms of the permutation is greater than .

Solution 1

If the first number is , then there are no restrictions. There are , or ways to place the other numbers. If the first number is , can go in four places, and there are ways to place the other numbers. ways. If the first number is , .... 4 6 \_ \_ \_ \_ 24 ways 4 \_ 6 \_ \_ \_ 24 ways 4 \_ \_ 6 \_ \_ 24 ways 4 \_ \_ \_ 6 \_ 5 must go between and , so there are ways. ways if 4 is first. If the first number is , .... 3 6 \_ \_ \_ \_ 24 ways 3 \_ 6 \_ \_ \_ 24 ways 3 1 \_ 6 \_ \_ 4 ways 3 2 \_ 6 \_ \_ 4 ways 3 4 \_ 6 \_ \_ 6 ways 3 5 \_ 6 \_ \_ 6 ways 3 5 \_ \_ 6 \_ 6 ways 3 \_ 5 \_ 6 \_ 6 ways 3 \_ \_ 5 6 \_ 4 ways ways If the first number is , .... 2 6 \_ \_ \_ \_ 24 ways 2 \_ 6 \_ \_ \_ 18 ways 2 3 \_ 6 \_ \_ 4 ways 2 4 \_ 6 \_ \_ 4 ways 2 4 \_ 6 \_ \_ 6 ways 2 5 \_ 6 \_ \_ 6 ways 2 5 \_ \_ 6 \_ 6 ways 2 \_ 5 \_ 6 \_ 4 ways 2 4 \_ 5 6 \_ 2 ways 2 3 4 5 6 1 1 way ways Grand Total :

Solution 2

If is the first number, then there are no restrictions. There are , or ways to place the other numbers. If is the second number, then the first number can be or , and there are ways to place the other numbers. ways. If is the third number, then we cannot have the following: 1 \_ 6 \_ \_ \_ 24 ways 2 1 6 \_ \_ \_ 6 ways ways If is the fourth number, then we cannot have the following: 1 \_ \_ 6 \_ \_ 24 ways 2 1 \_ 6 \_ \_ 6 ways 2 3 1 6 \_ \_ 2 ways 3 1 2 6 \_ \_ 2 ways 3 2 1 6 \_ \_ 2 ways ways If is the fifth number, then we cannot have the following: \_ \_ \_ \_ 6 5 24 ways 1 5 \_ \_ 6 \_ 6 ways 1 \_ 5 \_ 6 \_ 6 ways 2 1 5 \_ 6 \_ 2 ways 1 \_ \_ 5 6 \_ 6 ways 2 1 \_ 5 6 \_ 2 ways 2 3 1 5 6 4, 3 1 2 5 6 4, 3 2 1 5 6 4 3 ways ways Grand Total :

Solution 3

(needs explanation) The answer is .

Solution 4

(General Case, and you won't get 458, 459, 460, 462, 465, 467, etc. with this method!!!) First let us look at the General Case of this kind of Permutation: Consider this kind of Permutation of set for arbitrary It is easy to count the total number of the permutation () of : For every , we can divide into two subsets: Define permutation as the permutation satisfy the condition of this problem. Then according to the condition of this problem, for each , is not a permutation of set . For each , mark the number of permutation of set as , where , mark the number of permutation for set as ; then, according to the condition of this problem, the permutation for is unrestricted, so the number of the unrestricted permutation of is . As a result, for each , the total number of permutation is Notice that according to the condition of this problem, if you sum all up, you will get the total number of permutation of , that is, Put , we will have So the total number of permutations satisify this problem is .

## Problem 12

Let be a convex quadrilateral with , , and . Assume that the diagonals of intersect at point , and that the sum of the areas of triangles and equals the sum of the areas of triangles and . Find the area of quadrilateral .

Solution 1

For reference, , so is the longest of the four sides of . Let be the length of the altitude from to , and let be the length of the altitude from to . Then, the triangle area equation becomes . What an important finding! Note that the opposite sides and have equal length, and note that diagonal bisects diagonal . This is very similar to what happens if were a parallelogram with , so let's extend to point , such that is a parallelogram. In other words, and . Now, let's examine . Since , the triangle is isosceles, and . Note that in parallelogram , and are congruent, so and thus . Define , so . We use the Law of Cosines on and : Subtracting the second equation from the first yields This means that dropping an altitude from to some foot on gives and therefore . Seeing that , we conclude that is a 3-4-5 right triangle, so . Then, the area of is . Since , points and are equidistant from , so and hence .

Solution 2

(Another way to get the middle point) So, let the area of triangles , , , . Suppose and , then it is easy to show that . Also, because , we will have . So . So . So . So . As a result, . Then, we have . Combine the condition , we can find out that . So is the middle point of

## Problem 13

Misha rolls a standard, fair six-sided die until she rolls 1-2-3 in that order on three consecutive rolls. The probability that she will roll the die an odd number of times is where and are relatively prime positive integers. Find .

Solution 1

Let , with the subscript indicating an odd number of rolls. Then . The ratio of is just . We see that is the sum of ,,,... , while is the sum of , , ,... . , the probability of getting rolls of 1-2-3 in exactly 3 rolls, is obviously . We set this probability of aside, meaning we totally remove the chance of getting 1-2-3 in 3 rolls. Now the ratio of to should be equal to the ratio of , because in this case the 1st roll no longer matters, so we can disregard the very existence of it in counting how many times of rolls, and thus, 4 rolls, 6 rolls, 8 rolls... would become an odd number of rolls (while 5 rolls, 7 rolls, 9 rolls... would become even number of rolls). Notice , and also So we have . Finally, we get . Therefore, .

Solution 2

Call the probability you win on a certain toss , where is the toss number. Obviously, since the sequence has length 3, and . Additionally, . We can call this value , to keep our further equations looking clean. We can now write our general form for as . This factors the probability of the last 3 rolls being 1-2-3, and the important probability that the sequence has not been rolled in the past (because then the game would already be over). Note that since you'll win at some point. An intermediate step here is figuring out . This is equal to . Adding up all the differences, i.e. will give us the amount by which the odds probability exceeds the even probability. Since they sum to 1, that means the odds probability will be half of the difference above one-half. Subbing in our earlier result from the intermediate step, the odd probability is equal to . Another way to find the odd probability is simply summing it up, which turns out to be . Note the infinite sums in both expressions are equal; let's call it . Equating them gives , or . Finally, substituting , we find that , giving us a final answer of . --DanDan0101

Solution 3

Let be the number of strings of length containing the digits through that do not contain the string . Then we have because we can add any digit to end of a string with length but we have to subtract all the strings that end in . We rewrite this as We wish to compute since the last three rolls are for the game to end. Summing over the recursion, we obtain

Now shift the indices so that the inside term is the same:

Note that and . Therefore, Solving for , we obtain .

Solution 4

Let . is a transition matrix for the prefix of 1-2-3 matched so far. The state corresponding to a complete match has no outgoing probability mass. The probability that we roll the dice exactly times is . Thus the probability that we roll the dice an odd number of times is . Thus the answer is .

## Problem 14

The incircle of triangle is tangent to at . Let be the other intersection of with . Points and lie on and , respectively, so that is tangent to at . Assume that , , , and , where and are relatively prime positive integers. Find .

Solution 1

Let sides and be tangent to at and , respectively. Let and . Because and are both tangent to and and subtend the same arc of , it follows that . By equal tangents, . Applying the Law of Sines to yields Similarly, applying the Law of Sines to gives It follows that implying . Applying the same argument to yields from which . The requested sum is .

Solution 2

(Projective) Let the incircle of be tangent to and at and . By Brianchon's theorem on tangential hexagons and , we know that and are concurrent at a point . Let . Then by La Hire's lies on the polar of so lies on the polar of . Therefore, also passes through . Then projecting through , we have Therefore, . Since we know that and . Therefore, and . Since , we also have . Solving for , we obtain .

Solution 3

(Combination of Law of Sine and Law of Cosine) Let the center of the incircle of be . Link and . Then we have Let the incircle of be tangent to and at and , let and . Use Law of Sine in and , we have therefore we have Solve this equation, we have As a result, , , , , So, Use Law of Cosine in and , we have And we have So Solve this equation, we have As a result, So, the final answer of this question is

## Problem 15

Find the number of functions from to the integers such that , , and for all and in .

Solution

First, suppose . Then, the inequality becomes . In other words, the (positive) difference between consecutive function values is , , or . Let . Note that Thus, . Note that at most one value of in can be negative. This is because the maximum value of would be if more than one value of is negative. Plugging into the original inequality yields , which becomes . The only way for to be negative while satisfying this inequality is for to equal or . However, this forces , which is disallowed. Hence, we conclude that the following stronger inequality, is always true. We now have two cases of functions to count. For future reference let be the (ordered) sequence . Case 1: There is exactly one instance of . By the "stronger" inequality above, if , and if . If , then contains the subsequence , and the other three -values sum to , so they are either , , and in some order, or they are , , and in some order. Thus, each for which produces sequences . If or , then begins with or ends with , respectively. Either way, the remaining four -values sum to , so they can be any permutation of (six permutations) or (four permutations). Each of these vaues of yields sequences, so our total count for Case 1 is . Case 2: All values of are positive. Then, is a permutation of , , , or . The number of ways to permute three s and three s is . The number of ways to permute two s, two s, and two s is . The number of ways to permute one , four s, and one is . Finally, there is obviously only way to permute six s. Our total count for Case 2 is . To complete the justification that all of these cases satisfy the original inequality, we leverage the fact that is either monotonically increasing (Case 2) or the union of two monotonically increasing subsequences (Case 1). Consider any monotonically increasing subsequence that starts at and ends at . For , will be positive, allowing us to remove the absolute value bars from the original inequality: Now, the inequality is transitive; supposing , if the inequality is satisfied at and at , then it is also satisfied at . If we ever have a decreasing part where , then we can use some variant of the inequality , which we derived earlier. This is a specific case of , so we can finish off the argument by invoking transitivity.